ON THE RANGE OF RANDOM WALK

BY

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ABSTRACT

Let $\{S_n, n = 0, 1, 2, \ldots\}$ be a random walk (S_n being the nth partial sum of a sequence of independent, identically distributed, random variables) with values in E_d, the d-dimensional integer lattice. Let $f_n = \text{Prob}\{S_1 \neq 0, \ldots, S_{n-1} \neq 0, S_n = 0 \mid S_0 = 0\}$. The random walk is said to be transient if $p = 1 - \Sigma_{k=1}^{\infty} f_k > 0$ and strongly transient if $\Sigma_{n=1}^{\infty} \Sigma_{k=n+1}^{\infty} f_k < \infty$. Let $R_n =$ cardinality of the set $\{S_0, S_1, \ldots, S_n\}$. It is shown that for a strongly transient random walk with $p < 1$, the distribution of $[R_n - np]/\sigma\sqrt{n}$ converges to the normal distribution with mean 0 and variance 1 as n tends to infinity, where σ is an appropriate positive constant. The other main result concerns the "capacity" of $\{S_0, \ldots, S_n\}$. For a finite set A in E_d, let $C(A) = \Sigma_{x \in A} \text{Prob}\{S_n \notin A, n \geqq 1 \mid S_0 = x\}$ be the capacity of A. A strong law for $C\{S_0, \ldots, S_n\}$ is proved for a transient random walk, and some related questions are also considered.

0. **Introduction.** By a *random walk* we mean a stochastic process $(S_n, n = 0, 1, \cdots)$ which assumes its values in the d-dimensional space E_d of integer lattice points, such that $S_0 = 0$ and the increments $X_k = S_k - S_{k-1}$, $k = 1, 2, \cdots$ constitute a sequence of independent identically distributed random variables.

Let R_n be the cardinality of the random set $\{S_0, S_1, \cdots, S_n\}$. This random variable was studied by Dvoretzky and Erdös [1]. By *simple* random walk in d dimensions is meant the random walk which moves from the origin to any of its $2d$ neighbors with probability $(2d)^{-1}$. For such random walks it was shown in [1] that $R_n \sim E[R_n]$ when $d \geqq 2$, where $E[\,\cdot\,]$ is the expectation operator. In [3] Spitzer showed how the ergodic theorem can be used to prove that for arbitrary random walk $R_n/n \sim p$, where $p = P[S_1 \neq 0, S_2 \neq 0, \cdots]$. (In the transient case this includes the Dvoretzky-Erdös result, but in the more difficult case of 2-dimensional simple random walk more refined results are obtained in [1]). The first problem considered by us is that of a limiting distribution for R_n, with suitable normalization.

Let $f_n = P[S_1 \neq 0, S_2 \neq 0, \cdots, S_{n-1} \neq 0, S_n = 0]$. It is usual to call a random walk *transient* if $\sum_{k=1}^{\infty} f_k < 1$. Let $r_n = \sum_{k=n+1}^{\infty} f_k$, $n = 0, 1, \cdots$. Following

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[2] the random walk will be called *strongly transient* if $\sum_{k=0}^{\infty} r_k < \infty$. We let $t_n = \sum_{k=n+1}^{\infty} r_k$. We are obliged to Professor Port for pointing out the relevance of strong transience in the present context. In section 1 it is proved that in the strongly transient case $p \neq 1$ implies that $(R_n - pn)/\sqrt{n}$ asymptotically has a non-degenerate normal distribution. Since the assumption of strong transience entails a considerable amount of asymptotic independence for the sequence of differences $(R_{n+1} - R_n)$ the conclusion is hardly surprising. However, none of the numerous limit theorems for dependent random variables now in the literature seems to serve to give an easy proof of our result.

Occasionally it will be useful to consider the random walk started at a point x which may be different from 0. In that case the probability of an event will be denoted by $P_x[\cdot]$ instead of the ordinary $P[\cdot]$, e.g. $P[S_0 = 0] = 1$ and $P_x[S_0 = x] = 1$. Corresponding to any transient random walk there is a capacity $C(\cdot)$. To define this first introduce

$$\Psi_x(A) = \begin{cases} P_x[S_1 \notin A, S_2 \notin A, \cdots], & x \in A \\ 0 \qquad\qquad\qquad\qquad , & x \notin A, \end{cases}$$

the probability of *escaping from A starting at* x, then set $C(A) = \sum_{x \in A} \Psi_x(A)$. For an extensive discussion see [3]. In Section 2 we prove a strong law for $C_n = C(\{S_0, S_1, \cdots, S_n\})$ and discuss some related questions.

If B is an event the *indicator* of B, denoted by I_B, is the function such that $I_B(\omega) = 1$ (0) if $\omega \in B (\omega \notin B)$.

1. **Central limit theorem for** R_n. Let Z_k be the indicator of $[S_k \neq S_{k-1}, S_k \neq S_{k-2}, \cdots, S_k \neq 0]$, so that Z_k equals 1 if the random walk visits a new point at time k, 0 otherwise. Then

$$R_n = \sum_{k=0}^{n} Z_k.$$

Reversing the time direction of the path $(S_0, S_1, \cdots, S_n)$ one sees that $P[Z_n = 1] = P[S_1 \neq 0, S_2 \neq 0, \cdots, S_n \neq 0]$, so that Z_n has the same distribution as W_0^n, the indicator of the event [not returning to 0 by time n]. More generally, if W_j^n is the indicator of $[S_{j+1} \neq S_j, \ S_{j+2} \neq S_j, \cdots, S_n \neq S_j]$ one sees that $(Z_n, Z_{n-1}, \cdots, Z_1, Z_0)$ has the same distribution as $(W_0^n, W_1^n, \cdots, W_{n-1}^n, W_n^n)$. This observation was made in [1], and it will serve as well below.

Let $p_j = P[Z_j = 1]$, $p_{jk} = P[Z_j = 1, Z_k = 1]$. Note that as $j \to \infty$, $p_j \downarrow p$. Since $p_j = P[W_0^j = 1]$, $p = P[S_i \neq 0, \ i = 1, 2, \cdots]$. Let σ_n^2 be the variance of R_n. We shall write $a_n \sim b_n$ to mean that $a_n / b_n \to 1$ as $n \to \cdots$.

LEMMA. *Let* (S_n) *be strongly transient,* $p < 1$. *Then there exists a positive constant* σ^2 *such that* $\sigma_n^2 \sim \sigma^2 n$ *as* $n \to \infty$.

Proof. Evidently

$$\sigma_n^2 = \sum_{k=0}^{n} p_k(1-p_k) + 2\sum_{j=0}^{n-1}\sum_{k=j+1}^{n}(p_{jk}-p_jp_k) \tag{1.1}$$

and since

$$\sum_{k=0}^{n} p_k(1-p_k) \sim p(1-p)n \tag{1.2}$$

only the last term in (1.1) needs examination.

For $j < k$ one obtains

$$p_{jk} = p_jp_{k-j} - P[W_0^{k-j}=1,\ W_{k-j}^{k}=1,\ W_0^{k}=0]$$

so that on setting

$$a_{jk} = P[W_0^{k-j}=1,\ W_{k-j}^{k}=1,\ W_0^{k}=0]$$

the second term on the right side of (1.1) reduces to two times

$$\sum_{j=0}^{n}\sum_{k=j+1}^{n}(p_jp_{k-j}-p_jp_k) - \sum_{k=1}^{n}\sum_{j=0}^{k-1}a_{jk}.$$

Since

$$p_j = p + r_j$$

one obtains

$$\sum_{j=0}^{n}\sum_{k=j+1}^{n}(p_jp_{k-j}-p_jp_k) = \sum_{j=0}^{n}\sum_{k=j+1}^{n}p_j(r_{k-j}-r_k) \sim$$
$$\sum_{j=0}^{n} p\left(\sum_{i=1}^{n-j}r_i - \sum_{i=j+1}^{n}r_i\right) \sim pt_0n.$$

It will be shown that there exists a constant a such that

$$\sum_{k=0}^{n}\sum_{j=0}^{k-1}a_{jk} \sim an$$

by proving that

$$\sum_{j=0}^{k} a_{jk} \to a < \infty \text{ as } \to \infty \tag{1.3}$$

and the above relations substituted into (1.1) imply

$$\sigma_n^2 \sim (p(1-p) + 2(pt_0 - a))n. \tag{1.4}$$

Let $W_j^{\infty} = \lim_{n\to\infty} W_j^n$. To prove (1.3) write

$$(1.5)\qquad \sum_{j=0}^{k} a_{j,k} = \sum_{j=0}^{k} a_{k-j,k} = \sum_{j=0}^{k} P[W_0^j = 1, W_j^k = 1, W_0^k = 0]$$

$$\to \sum_{j=0}^{\infty} P[W_0^j = 1, W_j^\infty = 1, W_0^\infty = 0] = a,$$

where the limit is taken as $k \to \infty$ and the justification of the limit assertion follows from the dominated convergence theorem using the evident fact that $a_{k-j,k} \leqq r_j$. This proves (1.4). It must now be shown that the coefficient of n in (1.4) is positive. In (1.5) there is an explicit expression for a, and we claim that it satisfies

$$(1.6)\qquad \sum_{j=0}^{\infty} P[W_0^\infty = 1, W_j^\infty = 1, W_0^\infty = 0] = \sum_{j=0}^{\infty} P[W_0^j = 1, W_0^\infty = 0] \cdot P[W_j^\infty = 1 \mid W_0^j = 1, W_0^\infty = 0]$$

$$\leqq \sum_{j=0}^{\infty} P[W_0^j = 1, W_0^\infty = 0] P[W_j^\infty = 1] = t_0 p$$

and it is only the inequality which is not evident. The desired inequality will be justified if it is shown that for each j, $P[W_j^\infty = 1 \mid W_0^j = 1, W_0^\infty = 0] \leqq P[W_j^\infty = 1]$, and this will follow if it can be shown that for every x different from 0, P_x [never returning to $x \mid 0$ is visited some time] $\leqq P_x$ [never returning to x]. Let $R =$ [returning to x some time], $V =$ [visiting 0 some time], and let V^c be the complement of V. The desired inequality is equivalent to $P_x[R \mid V] \geqq P_x[R \mid V^c]$. To prove this let $R_0 =$ [x is revisited some time but 0 is not visited before the first return to x], $R_1 = R - R_0$. The desired inequality then follows from

$$P_x[V^c]P_x[R \cap V] = P_x[V^c](P_x[R_0]P_x[V] + P_x[R_1])$$

$$\geqq P_x[V]P_x[R_0]P_x[V^c] = P_x[V]P_x[R \cap V^c].$$

THEOREM 1. *Let (S_n) be strongly transient, $p < 1$. Then the distribution of $(R_n - np)/\sigma\sqrt{n}$ converges to the normal distribution with mean 0, variance 1.*

Proof. Given positive integers m', m'', $m = m' + m''$ let $\Delta_h = \{j : (h-1)m \leqq j < hm\}$, $h = 1, 2, \cdots$. Let Δ_h' consist of the first m' members of Δ_h', and let $\Delta_h'' = \Delta_h - \Delta_h'$ consist of the remaining m'' elements of Δ_h. Write $[n/m]$ for the greatest integer in n/m. Let $D_n = \cup \{\Delta_h : 1 \leqq h \leqq [n/m]\}$, $D_n'' = \cup \{\Delta_h'' : 1 \leqq h \leqq [n/m]\}$, $D_n''' = \{j : m[n/m] \leqq j \leqq n\}$.

Now random variables U_i, V_i will be introduced, $i = 0, 1, \cdots$. If $i \in \Delta_h$, let U_i, be the indicator of $[S_i \neq S_{i-1}, S_i \neq S_{i-2}, \cdots, S_i \neq S_{(h-1)m}]$ and let $V_i = U_i - Z_i$. Note that U_i and V_i depend on m' and m'', and that V_i is also an indicator. It is possible to choose m' and m'' as functions of n in such a manner that they tend to infinity with n and the following conditions are satisfied:

(1.7) $$n^{1/2} t_{m'}/m \to 0$$

(1.8) $$m/n^{1/2} \to 0$$

(1.9) $$m'/m \to 0.$$

Suppose $i \in \Delta_h$, or more specifically

$$i = (h-1)m + j, \qquad 0 \leqq j < m.$$

Then $P[V_i = 1] \leqq r_j$ and therefore

$$\sum_{h=1}^{[n/m]} \sum_{i \in \Delta_h''} E[V_i] \leqq \sum_{h=1}^{[n/m]} \sum_{j=m'}^{m} r_j \leqq t_{m'}[n/m].$$

Thus for any $\varepsilon > 0$

$$\varepsilon n^{1/2} P\left[\sum_{i \in D_n''} V_i \geqq \varepsilon n^{1/2} \right] \leqq \sum_{i \in D_n''} E[V_i] \leqq [n/m] t_{m'}$$

and using (1.7) one obtains that as $n \to \infty$

(1.10) $$P\left[n^{-1/2} \sum_{i \in D_n''} V_i \geqq \varepsilon \right] \leqq \varepsilon^{-1}(n^{1/2}/m) t_{m'} \to 0.$$

Observe now that

(1.11) $$\frac{1}{\sigma\sqrt{n}} \sum_{i \in D' \cup D''} (U_i - E[U_i]) = \sum_{h=1}^{[n/m]} \left\{ \frac{1}{\sigma\sqrt{n}} \sum_{i \in \Delta_h} (U_i - E[U_i]) \right\} = \sum_{h=1}^{[n/m]} X_{nh}$$

is the sum of $[n/m]$ independent and identically distributed random variables. The sum has mean 0 and by the Lemma the variance of the sum tends to 1 as $n \to \infty$. The Lindeberg condition for the central limit theorem is that for every $\varepsilon > 0$

(1.12) $$\sum_{h=1}^{[n/m]} \int_{|X_{nh}| \geqq \varepsilon} X_{nh}^2 \to 0 \text{ as } n \to \infty.$$

Now $|X_{nh}| \leqq m/(\sigma\sqrt{n})$, so that condition (1.8) insures the truth of (1.12) and so the sums in (1.11) asymptotically have a normal distribution with mean 0, variance 1. It follows from the Lemma that the sum $\sum_{i \in D_n'}(U_i - E[U_i])$ has a variance asymptotically equal to $n\sigma^2 m'/m$. By (1.9) this variance is then $o(n)$ and so $n^{-1/2}\sum_{i \in D_n'}(U_i - E[U_i])$ converges to 0 in probability, and one may conclude that the distribution of $(\sigma\sqrt{n})^{-1}\sum_{i \in D_n''}(U_i - E[U_i])$ converges to the normal with mean 0, variance 1. In view of (1.10) the same conclusion holds for the distribution of the sum $(\sigma\sqrt{n})^{-1}\sum_{i \in D_n''}(Z_i - E(Z_i)]$. It only remains to verify that $n^{-1/2}\sum_{i \in D_n'''}(Z_i - E[Z_i])$ and $n^{-1/2}\sum_{i \in D_n'}(Z_i - E[Z_i])$ converge to 0 in probability. In view of (1.8) and the fact that D_n''' contains at most m indices this is evident for the first sum. Proceeding as in the second paragraph of the proof of the Lemma one obtains

$$\sum_{k=j+1}^{n} (p_{jk} - p_j p_k) \leqq p_j \sum_{i=1}^{n-j} r_i \leqq p_1 t_0$$

so that the variance of $\sum_{i \in D_n'} Z_i$ is bounded by some constant times the number of terms in D_n', that is, it is $0(nm'/m)$, and hence $o(n)$ by (1.9), which suffices for our purpose.

If the random walk has mean 0 and finite second moment, then it is strongly transient if and only if its genuine dimension is greater than or equal to 5. Thus Theorem 1 does not apply if, for example, the genuine dimension is 3. However, using these methods and the crude estimate for the variance of R_n that is given in [1], we can show that for $\alpha > 2/3$, and $\varepsilon > 0$,

$$P\{|R_n - ER_n| > \varepsilon n^\alpha\} \to 0$$

as $n \to \infty$.

2. **Strong law for C_n.** Corresponding to any transient random walk $\{S_i\}$ there is a capacity $C(\cdot)$ as explained in the introduction. Let $C_n = C(\{S_0, S_1, \cdots, S_n\})$. Let Z_k have the same significance as above and set $Z_{n,k} = Z_k \cdot \Psi_{S_k}(\{S_0, S_1, \cdots, S_n\})$. Then $C_n = \sum_{k=0}^{n} Z_{n,k}$.

In [3] an ergodic theorem was used to obtain a strong law for R_n; we employ it here to derive a strong law for C_n.

Let $e_{n,k} = E[Z_{n,k}]$. For fixed k, $Z_{n,k}$ decreases as n increases and so as n approaches infinity $e_{n,k}$ decreases to e_k. For $j < k$, $Z_{n,j}$ has the same distribution as

$$I_{[S_k \neq S_{k-1}, S_k \neq S_{k-2}, \cdots,\ S_k \neq S_{k-j}]} \Psi_{S_k}(\{S_{k-j}, S_{k-j+1}, \cdots, S_{n+k-j}\})$$

and this last quantity is greater than or equal to $Z_{n+k-j,k}$. Hence $e_{n,j} \geqq e_{n+k-j,k}$ and therefore $e_j > e_k$ and as n approaches infinity e_n decreases to a limit e_∞.

The random walk $\{S_i\}$ takes its values in the d-dimensional space of the integer lattice points E_d. Introduce the probability space

$$(\Omega, \mathscr{B}, P) = \prod_{-\infty < i < \infty} (\Omega_i, \mathscr{B}_i, P_i)$$

where $\Omega_i = E_d$, $\mathscr{B}_i$ consists of all subsets of E_d, and $P_i(B) = P[S_1 \in B]$. The elements of Ω are doubly-infinite sequences, $\omega = (x_i)$, $-\infty < i < \infty$. The *shift* T is defined by $T(x_i) = (y_i)$, $y_i = x_{i+1}$. For $\omega = (x_i)$ let

$$S_0(\omega) = 0,\ S_n(\omega) = x_1 + x_2 + \cdots + x_n,\ S_{-n}(\omega) = -(x_0 + x_1 + \cdots + x_{-n+1}),$$
$$n = 1, 2, \cdots.$$

This gives rise to a process with stationary independent increments and $(S_0, S_1, \cdots)$ can be taken to be the original random walk.

Let

$$Z = I_{[S_{-1} \neq 0, S_{-2} \neq 0, \cdots]}, Y = \Psi_{S_0}(\{\cdots S_{-1}, S_0, S_1, \cdots\})$$

and let $W = ZY$.

The following two relations are easily checked:

$$\lim_{m \to \infty} \frac{1}{m}(e_{m1} + e_{m2} + \cdots + e_{mm}) = e_\infty \tag{2.1}$$

$$E[W] = e_\infty. \tag{2.2}$$

The ergodic theorem applies to give

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n} W(T^k \omega) = E[W] = e_\infty$$

with probability one. Since $W(T^k\omega) \leqq Z_{n,k}(\omega)$ it follows that

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} Z_{nk} \geqq e_\infty. \tag{2.3}$$

To obtain an inequality in the opposite direction let m be a positive integer and set $C_h^{(m)} = C(\{S_{mh}, S_{mh+1}, \cdots, S_{m(h+1)-1}\})$. Observe that the $C_h^{(m)}$, $h = 0, 1, \cdots$ are independent and identically distributed, so the Kolmogorov strong law gives as $n \to \infty$

$$\frac{1}{n} \sum_{h=0}^{[n/m]+1} C_h^{(m)} \to \frac{1}{m} E[C_0^{(m)}] = \frac{1}{m} \sum_{i=0}^{m-1} e_{mi}$$

with probability one. The term on the left dominates $n^{-1}C_n$, and applying (2.1) to the term on the right gives

$$\limsup_{n \to \infty} n^{-1} C_n \leqq e_\infty. \tag{2.4}$$

Together (2.3) and (2.4) imply

THEOREM 2. *As $n \to \infty$, $n^{-1}C_n \to e_\infty$ with probability one.*

It becomes of interest to know when $e_\infty > 0$. Call a subset A of the state space *recurrent* if $P[S_i \in A$ for infinitely many $i] = 1$. The proofs of the following facts are easy:

(i) $e_\infty = 0$ is equivalent to $EY = 0$ and also to $\{\cdots S_{-1}, S_0, S_1, \cdots\}$ being a recurrent set with probability 1.

(ii) For random walk with mean 0 and finite variance the condition $\{\cdots S_{-1}, S_0, S_1, \cdots\}$ recurrent with probability one is equivalent to $\{S_0, S_1, \cdots\}$ recurrent with probability one and also to $\{\cdots S_{-1}, S_0\}$ recurrent with probability one.

(iii) Random walk with mean 0, finite variance is strongly transient if and only if it is genuinely d-dimensional, with $d \geqq 5$.

(iv) For strongly transient random walk with mean 0 and finite variance $e_\infty > 0$.

Finally a converse to (iv) will be established

(v) For random walk with mean 0, and finite variance in less than or equal to 4 dimensions $e_\infty = 0$.

Proof. It suffices to consider the 4-dimensional case. Let $B_\rho = \{x : |x| \leqq \rho\}$. Then $C(B_\rho) = 0(\rho^2)$ as $\rho \to \infty$. Indeed for simple random walk in 4 dimension $C(B_\rho) \sim c\rho^2$ for a positive c. Since for arbitrary 4-dimensional random walk with mean 0, finite variance there exists a constant c_1 such that $C(B) \leqq c_1$ (capacity of B with respect to simple random walk) (see Spitzer [3], p. 321), it follows that $C(B_\rho) = 0(\rho^2)$. However for any $\varepsilon > 0$, $P[\sup_{k \leqq n} |S_k| \leqq \varepsilon\sqrt{n}]$ is bounded away from 0. Hence $P[C_n \leqq \varepsilon n]$ is bounded away from 0, and this is compatible with Theorem 2 only if $e_\infty = 0$.

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